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The annihilation of a two-dimensional jet by a transverse magnetic field

By H. K. MOFFATT AND J. TOOMRE
Department of Applied Mathematics and Theoretical Physics,
Silver Street, Cambridge

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The effect of an applied transverse magnetic field on the development of a two-dimensional jet of incompressible fluid is examined. The jet is prescribed in terms of its mass flux ρQ_0 and its lateral scale d at an initial section $x = 0$. The three dimensionless numbers characterizing the problem are a Reynolds number $R = Q_0/\nu$, a magnetic Reynolds number $R_m = \mu\sigma Q_0$, and a magnetic interaction parameter $N = \sigma B_0^2 d^2/\rho Q_0$, where ρ represents density, σ conductivity, μ permeability and B_0 applied field strength, and it is assumed that

$$R_m \ll 1, \quad R \gg 1, \quad N \ll 1.$$

It is shown that when $M^2 = RN \gg 1$, an inviscid treatment is appropriate, and that the effect of the magnetic field is then to destroy the jet momentum within a distance of order N^{-1} in the downstream direction. A general solution for inviscid development is obtained, and it is shown that a large class of velocity profiles (though not all of them) are self-preserving.

When $M^2 \ll 1$, it is shown that the viscous similarity solution obtained by Moreau (1963*a, b*) is relevant. This solution is re-derived and re-interpreted; it implies that the jet momentum is destroyed within a distance of order $R^{\frac{1}{4}}N^{-\frac{3}{4}}$ in the downstream direction.

Some further aspects of the jet annihilation problem are qualitatively discussed in § 4, *viz.* the nature of the overall flow field, the effect of the presence of distant boundaries, the effect of increasing R_m to order unity and greater, and the effect of oblique injection. Finally the development of a jet of conducting fluid into a non-conducting environment is considered; in this case the jet is not stopped by the magnetic field unless a return path outside the fluid for the induced current is available.

1. Introduction

The effect of a transverse magnetic field on the development of a two-dimensional submerged jet of conducting fluid has been examined by Moreau (1963*a, b*). He obtained a similarity solution of the equations of magnetohydrodynamics (in boundary-layer approximation) representing a balance between inertia, magnetic and viscous forces. The solution revealed that the magnetic field tends to destroy the momentum of the jet and to cause it to diverge at a certain finite distance downstream from its point of origin. The same solution has been obtained by Tsinober & Shcherbinin (1965).

The solution is of great interest in that it is one of the very few in which fully non-linear effects compete with magnetic and viscous effects to establish the pattern of flow. However, the solution has certain limitations which have not as yet been fully appreciated. Like any similarity solution, the Moreau solution must be understood as being asymptotically valid at a large distance from the momentum source. However, the fact that the jet diverges at a certain finite distance downstream from the source means that there may not be sufficient space for the jet to settle down to its asymptotic form before it diverges. It will emerge from the analysis of this paper that the condition for the flow to settle down to Moreau's similarity form before the jet diverges is that the Hartmann number $M = (RN)^{\frac{1}{2}}$ should be small. If $M \gg 1$ then the Moreau solution is irrelevant, and an alternative description of the flow is required. It is the purpose of this paper to provide such a description. The analysis applies in particular to the inviscid limit $\nu \to 0$ (keeping all other parameters fixed).

Consider for definiteness the situation depicted in figure 1*a* (dimensional variables are distinguished by an asterisk). Incompressible conducting fluid fills the space $x^* > 0$, $|y^*| < y_0 d$, $|z^*| < z_0 d$, and a uniform magnetic field $\mathbf{B}_0 = (0, B_0, 0)$ is externally maintained. Fluid is steadily introduced across the boundary $x^* = 0$ with x^*-component of velocity $U(y^*/d)$; the y^*-component of velocity on $x^* = 0$ will for the moment remain unspecified. It will be supposed that $U(y^*/d)$ is summetrical about $y^* = 0$, has a single maximum U_0 at $y^* = 0$, and has a finite flux,†

$$2Q_0 = \int_{-\infty}^{\infty} U\,dy^* = 2U_0 d \quad \text{say,} \tag{1}$$

where d is the 'lateral scale' of $U(y^*/d)$. We suppose, moreover, that $y_0, z_0 \gg 1$; in fact we shall be interested in the nature of the flow at fixed values of the dimensionless variables $y = y^*/d$, $z = z^*/d$ in the limit $y_0, z_0 \to \infty$. It is to be expected that the (dimensionless) velocity field $\mathbf{u}(\mathbf{x})$ becomes two-dimensional in the limit $z_0 \to \infty$, i.e. that

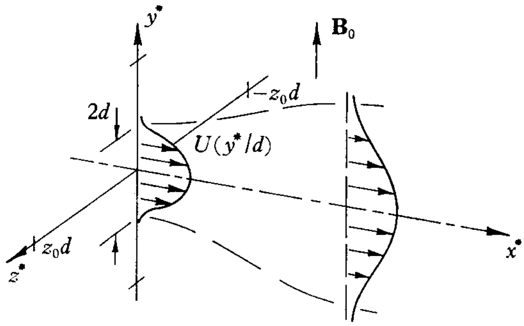
$$\mathbf{u} \approx (u(x,y), v(x,y)) = (\partial\psi/\partial y, -\partial\psi/\partial x), \tag{2}$$

where $Q_0\,\psi(x,y)$ is the stream-function of the flow. It is reasonable to suppose that $\psi(x,y)$ is insensitive to the nature of the boundaries $y = \pm y_0$, $z = \pm z_0$, e.g. as to whether they are porous or impermeable, electrically conducting or insulating, etc. It will appear below, however, that the pressure distribution *is* to some extent affected by the conditions on these distant boundaries. Possible effects due to finite values of y_0 will be considered in § 4.

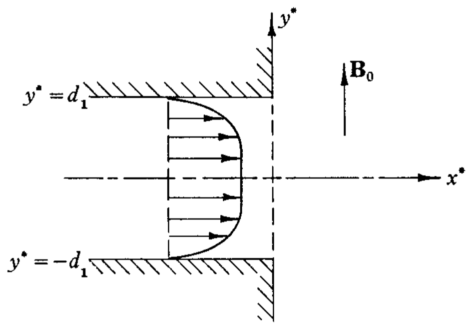
The prescribed condition on the boundary $x = 0$ is perhaps somewhat artificial, and requires comment. It may be imagined that the boundary $x = 0$ is a rigid sheet of material of variable porosity proportional to $U(y)$; if fluid is continually supplied to the region $x < 0$ at a pressure higher than the pressure at $x = 0+$, then the steady velocity profile $U(y)$ can (in principle) be maintained. In practice, the simplest possibility is that the fluid is supplied through a slit $|y^*| < d_1$ in the

† The momentum flux defined in (10) below is then also necessarily finite.

plane wall $x = 0$ (figure 1*b*). In this case, viscous forces may affect the velocity profile $U(y)$, although perhaps the most reasonable possibility is that the profile in the slit is an undeveloped 'top-hat' profile $U(y) = \text{const}$; this possibility is all the more reasonable if the Hartmann number based on d is large, since the well-known Hartmann profile for channel flow then has the same top-hat structure; ($d = d_1$ in this case).



(*a*)



(*b*)

FIGURE 1. Symmetrical injection of a two-dimensional jet into a region permeated by a uniform transverse magnetic field.

It will be assumed that

$$R_m = \mu\sigma Q_0 \ll 1, \tag{3}$$

where σ is the conductivity and μ the permeability of the fluid, and distortion of the magnetic field will be neglected (Shercliff 1965, § 3.8). (Qualitative effects of increasing R_m will be considered in § 4.3.)

The governing equations are then

$$\rho\mathbf{u}^*.\nabla^*\mathbf{u}^* = -\nabla^* p^* + \mathbf{j}^* \wedge \mathbf{B}_0 + \eta\nabla^{*2}\mathbf{u}^*, \tag{4}$$

$$\mathbf{j}^* = \sigma(\mathbf{E}^* + \mathbf{u}^* \wedge \mathbf{B}_0), \tag{5}$$

$$\nabla^*.\mathbf{u}^* = \nabla^*.\mathbf{j}^* = 0. \tag{6}$$

Moreover, under steady conditions, the electric field $\mathbf{E}^*$ is derivable from a potential, $\mathbf{E}^* = -\nabla^*\phi^*$, and equations (5) and (6) together imply that

$$\nabla^{*2}\phi^* = \nabla^*.(\mathbf{u}^* \wedge \mathbf{B}_0) = -\mathbf{B}_0.(\nabla^* \wedge \mathbf{u}^*). \tag{7}$$

Hence in the two-dimensional region of the flow (i.e. everywhere except near the distant boundaries $z = \pm z_0$), $\nabla^{*2}\phi^* = 0$. The electric field in the two-dimensional region is therefore unaffected by the flow in that region and is determined only by the conditions near the distant boundaries $z = \pm z_0$. Since the field $\mathbf{E}^*$ is independent of z far from these boundaries, we must have

$$\partial E_z^*/\partial x^* = \partial E_x^*/\partial z^* = 0, \quad \partial E_z^*/\partial y^* = \partial E_y^*/\partial z^* = 0,$$

so that $E_z^* =$ const. The value of E_z^* is determined by the electrical properties of the boundaries $z = \pm z_0$, $y = \pm y_0$ (see the last paragraph of this section).

In terms of the dimensionless stream function $\psi(x, y)$, the curl of (4) reduces to the dimensionless form

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} = N\frac{\partial^2\psi}{\partial y^2} - R^{-1}\nabla^4\psi, \tag{8}$$

where

$$R = Q_0/\nu, \quad N = \sigma B_0^2 d^2/\rho Q_0, \tag{9}$$

the Reynolds number and magnetic interaction parameter respectively.

If the flow in the region $x > 0$ is to have the character of a jet, then clearly the momentum flux at $x = 0$,

$$2F_0 = \int_{-\infty}^{\infty} U^2\, dy = kQ_0^2 d^{-1}, \quad \text{say}, \tag{10}$$

(where k is a constant of order unity characterizing the shape of the initial profile) must be sufficiently large for the inertia force to dominate (over a considerable part of the flow field) over both the viscous force and the magnetic drag, represented by the term $N\,\partial^2\psi/\partial y^2$ in (8). It will therefore be assumed† that

$$N \ll 1, \quad R \gg 1. \tag{11}$$

The conditions (3) and (11) are realistic for mercury jets in transverse fields (Moreau 1966*a*, *b*). Under these conditions, it is to be expected that the jet will have a 'long thin' character, at any rate for some distance downstream, and the usual boundary-layer approximation, $\nabla^2 \approx \partial^2/\partial y^2$ is legitimate; it may of course transpire that the approximation breaks down in certain regions of the flow field. Equation (8) then takes the approximate form

$$\frac{\partial(\psi, \partial^2\psi/\partial y^2)}{\partial(x,y)} = N\frac{\partial^2\psi}{\partial y^2} - R^{-1}\frac{\partial^4\psi}{\partial y^4}, \tag{12}$$

which may be integrated once with respect to y to give

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = N\frac{\partial\psi}{\partial y} - R^{-1}\frac{\partial^3\psi}{\partial y^3}. \tag{13}$$

The arbitrary function of x which appears on integrating has been set equal to zero, to conform with the conditions that $u\ (= \partial\psi/\partial y)$ and the derivatives $\partial u/\partial y$, $\partial^2 u/\partial y^2$ should vanish as $y \to \pm\infty$.‡

† Some comments on the nature of the flow if $N \gg 1$ are given in § 4.2.

‡ A sufficient condition would be $u = o(|y|^{-1})$ as $|y| \to \infty$; this condition is satisfied by the initial profile $U(y)$, and the subsequent analysis confirms that it is invariably satisfied by $u(x, y)$ $(x > 0)$.

Equation (13) is simply the x-component of equation (4) in the boundary-layer approximation, and it is evident that the vanishing of the arbitrary function of x outside the jet region is equivalent to the condition

$$\partial p^*/\partial x^* = -\sigma E_z^* B_0, \tag{14}$$

which follows likewise from (5) and (6) with $u = 0$; if $E_z^* \neq 0$, a pressure gradient $\partial p^*/\partial x^*$ is established to counterbalance the $\mathbf{j}^* \wedge \mathbf{B}_0$ force outside the jet. As mentioned above, the presence of an electric field has no effect on the velocity field, but it does affect the pressure distribution.

In terms of the velocity distribution u, equation (13) takes the form

$$(\mathbf{u}.\nabla)\,u = -Nu + R^{-1}\,\partial^2 u/\partial y^2. \tag{15}$$

It is easy to see that this implies that the jet momentum is destroyed by the magnetic field. For, integrating from $y = -\infty$ to $y = +\infty$ we get

$$\frac{d}{dx}\int_{-\infty}^{\infty} u^2\,dy = -N\int_{-\infty}^{\infty} u\,dy,$$

or

$$dF/dx = -NQ, \tag{16}$$

where F and Q are the (non-dimensionalized) momentum and flux at the section x. Thus, as long as $Q > 0$, F decreases monotonically with increasing x. This decrease is directly due to the magnetic drag experienced by each fluid particle; it is also affected indirectly by viscous forces, in that they can (through viscous entrainment) locally increase the value of Q and so accelerate the decay of F (see § 3). Incompressibility of course requires that the destruction of momentum must be accompanied by a spreading of the jet in the lateral directions; this spreading is symmetrical due to the assumed symmetry of conditions at $x = 0$. At some stage, the assumption $\partial/\partial y \gg \partial/\partial x$ must break down; however, provided N is sufficiently small, equation (16) suggests that the jet will retain its jet-like character for some considerable distance downstream.

The destruction of jet momentum raises the question of how the over-all momentum balance of the fluid and its boundaries and any external electric circuits is maintained. A detailed description would require a consideration of the precise effects of the boundaries $y = \pm y_0$, $z = \pm z_0$, with the attendant three-dimensional difficulties. However, the essential nature of the balance may be understood from the following simple arguments. The jet momentum is destroyed essentially because a current $j_z = \sigma u B_0$ is induced in the jet region, and the associated Lorentz force $-\sigma B_0^2 u$ is retarding. The current circuit must however be completed in some manner that is determined by the distant boundary conditions. The return path for the current may be either through the fluid outside the jet region (as must happen, for example, if all the distant boundaries are insulating), or through the boundaries if they are perfectly conducting. In the former case, there must exist an electric field E_z^* and an adverse pressure gradient given by (14); the momentum balance is then provided by the pressure distribution that must be applied on the plane $x = 0$ to maintain the flow. In the latter case the integrated Lorentz force in the jet is balanced by a net Lorentz force acting on the boundaries that conduct the return current. If the boundaries have finite conductivity then the return current flows partly through the fluid and partly through the boundaries.

2. Behaviour of an inviscid jet ($RN \gg 1$)

In the limit $\nu \to 0$, or $R \to \infty$, equation (15) becomes, in Lagrangian form,

$$Du/Dt \equiv (\mathbf{u}.\nabla)\, u = -Nu. \tag{17}$$

(It will appear in retrospect that the necessary condition for the neglect of viscous forces is $RN \gg 1$.) The solution is, evidently,

$$u(\mathbf{x}_0, t) = u(\mathbf{x}_0, t_0)\, e^{-N(t-t_0)}, \tag{18}$$

for the fluid particle which was at $\mathbf{x}_0$ at time t_0. The x co-ordinate of this fluid particle at time t is determined by

$$Dx/Dt = u(\mathbf{x}_0, t) = u(\mathbf{x}_0, t_0)\, e^{-N(t-t_0)},$$

so that

$$x - x_0 = N^{-1} u(\mathbf{x}_0, t_0)\, (1 - e^{-N(t-t_0)}). \tag{19}$$

Hence, as $t \to \infty$, a fluid particle initially on $x = 0$ asymptotically approaches the line

$$x = N^{-1}\, u(0, y_0). \tag{20†}$$

Since, by definition, ψ is constant on streamlines, it follows that

$$\psi(x, y) = \psi(0, y_0);$$

hence (18) and (19) may be written in the Eulerian form

$$u(x, y) = u_0(\psi) - Nx, \tag{21}$$

where $u_0(\psi) = u(0, y_0)$. Since

$$u = \partial\psi/\partial y, \quad \text{so that} \quad \psi = \int_0^y u\, dy, \tag{22}$$

$u_0(\psi)$ may be regarded as known in principle (in its range of definition $|\psi| < 1$) if $u(0, y) = U(y)$ is known. Then from (21),

$$\int_0^\psi \frac{d\psi}{u_0(\psi) - Nx} = y, \tag{23}$$

and this implicitly determines $\psi(x, y)$.

The following examples of possible initial profiles will help to make the meaning of the above solution clear.

(i) First, suppose that the jet has a top-hat profile at $x = 0$, i.e. in non-dimensional form,

$$u(0, y) = \begin{Bmatrix} 1 & (|y| < 1), \\ 0 & (|y| > 1). \end{Bmatrix} \tag{24}$$

† The analysis is invalid if the initial profile has more than one maximum; for in this case there exist points $(0, y_1)$ and $(0, y_2)$ with $0 < y_1 < y_2$ for which $u(0, y_2) > u(0, y_1)$, and according to (20) the streamlines through the points $(0, y_1)$ and $(0, y_2)$ would intersect. Lateral pressure gradients, represented by the term $\partial p/\partial y$, must prevent this unphysical behaviour; in other words, the boundary-layer assumption $\partial/\partial y \gg \partial/\partial x$ is necessarily violated.

Then, for $y > 0$, on $x = 0$,

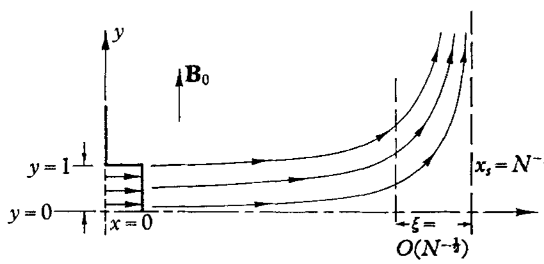
$$\psi = \begin{cases} y & (y<1), \\ 1 & (y>1). \end{cases} \tag{25}$$

Hence $$u_0(\psi) = 1 \quad (|\psi| < 1),$$
and from (21),

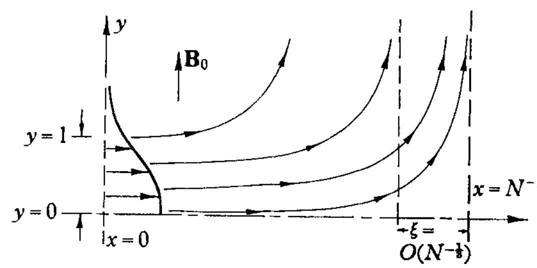
$$\partial\psi/\partial y = u(x,y) = 1 - Nx \quad (|\psi|<1),$$

i.e. $$\psi(x,y) = \begin{cases} (1-Nx)\,y & (0<y<(1-Nx)^{-1}), \\ 1 & (y>(1-Nx)^{-1}), \end{cases} \tag{26}$$

and of course, $\psi(x,-y) = -\psi(x,y)$.



(a)



(b)

FIGURE 2. Annihilation of an inviscid jet by a transverse magnetic field. The flow is symmetric about $y = 0$, and only the region $y > 0$ is sketched: (a) top-hat profile at $x = 0$; (b) sech^2 profile at $x = 0$. Boundary-layer theory breaks down between the broken lines in both cases.

The flow is sketched in figure 2a. The flow in the jet region has the same streamlines as the inviscid flow near a stagnation point on a rigid boundary. Outside the jet, i.e. for $|y| > (1-Nx)^{-1}$, the fluid is at rest. The magnetic field has an effect similar to that of a rigid boundary placed at $x = N^{-1}$. It would be misleading, however, to press the analogy too far; the streamlines within the jet are identical with the streamlines in a stagnation point flow, but they are not identical with the streamlines of the flow in an inviscid jet impinging normally on a rigid wall.

The transverse velocity $v = -\partial\psi/\partial x$ is given, from (26), by

$$v(x,y) = \begin{cases} Ny & (|y| < (1-Nx)^{-1}), \\ 0 & (|y| > (1-Nx)^{-1}). \end{cases} \tag{27}$$

Two observations may be made. First, within the limits of an inviscid analysis, v cannot be prescribed arbitrarily (in addition to u) on $x = 0$; it is determined by the solution (27). If the fluid is constrained to have a distribution of $v(0,y)$ inconsistent with (27), (e.g. by the use of guide vanes at the slit) then presumably there must be a viscous boundary-layer on the plane $x = 0$. If the 'natural' condition $v(0,y) = 0$ is imposed, then this layer is weak when $N \ll 1$.

Secondly, it is clear that the boundary-layer approximation ($\partial/\partial x \ll \partial/\partial y$), on which the solution (26) is based, breaks down at values of x where v becomes comparable in magnitude with u. The maximum value of v at a section x is $N(1-Nx)^{-1}$, and this is of the same order of magnitude as $u(x,0) = 1-Nx$ when $x = N^{-1}-\xi$ where $\xi = O(N^{-\frac{1}{2}})$. The above analysis is therefore valid only for

$$1-Nx \gg N^{\frac{1}{2}}; \tag{28}$$

in particular, it tells us nothing about the nature of the flow for $x \gtrsim N^{-1}$. The condition $N \ll 1$ may now be seen to be an essential prerequisite for the use of boundary-layer methods.

The breakdown of boundary-layer theory at $1-Nx = O(N^{\frac{1}{2}})$ is associated with the growing importance of pressure forces as this region is approached. The pressure distribution that is implied by the solution (26) may be obtained by integrating the y-component of (4), and in non-dimensional form this gives, within the jet,

$$p+Ex = -\tfrac{1}{2}N^2[y^2-(1-Nx)^{-2}]+\text{const.},$$

where $E = (\sigma B_0 d^3/\rho Q_0^2)E_z^*$ and $p = (d^2/\rho Q_0^2)p^*$. Hence

$$\partial p/\partial x + E = N^3(1-Nx)^{-3}.$$

The term on the right is neglected in the boundary-layer approximation. Its actual effect is probably to cause the jet to diverge at a value of x a little less than the value $x = N^{-1}$ suggested by boundary-layer theory.

(ii) Suppose now that the initial profile is

$$u(0,y) = \operatorname{sech}^2 y, \quad \psi(0,y) = \tanh y, \tag{29}$$

so that

$$u_0(\psi) = 1-\psi^2. \tag{30}$$

Equation (23) then becomes

$$y = \int_0^\psi \frac{d\psi}{1-Nx-\psi^2} = (1-Nx)^{-\frac{1}{2}}\tanh^{-1}\frac{\psi}{(1-Nx)^{\frac{1}{2}}},$$

i.e.

$$\psi(x,y) = (1-Nx)^{\frac{1}{2}}\tanh(1-Nx)^{\frac{1}{2}}y. \tag{31}$$

Hence

$$u(x,y) = \partial\psi/\partial y = (1-Nx)\operatorname{sech}^2(1-Nx)^{\frac{1}{2}}y. \tag{32}$$

The streamlines are sketched in figure 2*b*. In this case each streamline has a different asymptote, the streamline through $(0, y_0)$ having the asymptote

$$x = N^{-1}\operatorname{sech}^2 y_0.$$

The slower moving fluid near the outer edges of the jet is stopped at an earlier stage than the faster moving fluid at the centre. The flux $2Q = \psi(x,\infty)-\psi(x,-\infty)$ decreases continuously from 2 at $x = 0$ to zero at $x = N^{-1}$.

The maximum value of $|v|$ at the section x is in this case of order $N(1-Nx)^{-\frac{1}{2}}$, and this becomes of the same order as $u(x,0) = 1-Nx$ where $1-Nx = O(N^{\frac{2}{3}})$. Boundary-layer theory therefore in this case breaks down at a distance $O(N^{-\frac{1}{3}})$ short of the stopping plane $x = N^{-1}$.

(iii) In the examples (i) and (ii) given above, the velocity profiles are self-preserving in the sense that in both cases it is possible to express $\psi(x,y)$ in the form

$$\psi(x,y) = q(x)f(\eta), \quad \eta = y/\delta(x). \tag{33}$$

In case (i), $q = 1$, $\delta = (1-Nx)^{-1}$, and

$$f(\eta) = \left\{\begin{array}{ll} \eta & (|\eta| < 1), \\ 1 & (\eta > 1), \\ -1 & (\eta < -1), \end{array}\right\} \tag{34}$$

while in case (ii), $q = (1-Nx)^{\frac{1}{2}}$, $\delta = (1-Nx)^{-\frac{1}{2}}$, and $f(\eta) = \tanh\eta$. It is easily verified, on inspection of (23), that any initial profile of the form

$$u_0(\psi) = 1-|\psi|^p \quad (|\psi| < 1, p \geqslant 0), \tag{35}$$

is likewise self-preserving.†

Not every initial profile however behaves in this simple way. Consider, for example, the initial profile

$$u(0,y) = \frac{1}{1+(\pi y/2)^2}, \quad \psi(0,y) = \frac{2}{\pi}\tan^{-1}\frac{\pi y}{2}. \tag{36}$$

For this case, $u_0(\psi) = \cos^2(\pi\psi/2)$, and integration of (23) leads to the stream-function

$$\psi(x,y) = \frac{2}{\pi}\tan^{-1}\left\{\left(\frac{1-Nx}{Nx}\right)^{\frac{1}{2}}\tanh\frac{\pi y}{2}[Nx(1-Nx)]^{\frac{1}{2}}\right\}, \tag{37}$$

which cannot be expressed in the form (33).

The neglect of viscous forces is justified only if they cause negligible change in the jet profile over the distance N^{-1} to the stopping plane. In fact, viscous effects diffuse across the width of the velocity profile within a distance of order R downstream from the initial plane $x = 0$; hence the inviscid analysis of this section is reasonable provided $R \gg N^{-1}$, i.e. provided

$$M^2 = RN = \sigma B_0^2 d^2/\rho\nu \gg 1. \tag{38}$$

If this condition is satisfied, then the jet is stopped long before viscous diffusion has any significant effect.

3. The effect of viscous entrainment when $M^2 = RN \ll 1$

Suppose now instead that $M^2 \ll 1$. Then the magnetic force may be expected to have negligible influence over a range $x \lesssim O(R)$ over which viscous diffusion thoroughly redistributes the initial distribution of momentum in the jet. In the range in which x is large compared with R but still small enough for magnetic forces to be negligible (see (51) below), the well-known similarity solution

† In fact, by substitution of (33) into (13) (with $R = \infty$), it may be shown that (35) is also a necessary condition for self-preserving flows.

(Bickley 1937; Goldstein 1938, §57) may be expected to apply. This solution is usually expressed in terms of the jet momentum $2F_0$ defined in (10). In terms of the dimensionless variables used in this paper, Bickley's solution may be written

$$\psi(x,y) = \left[\frac{9k(x+x_0)}{2R}\right]^{\frac{1}{3}} \tanh\left[\frac{kR^2}{48(x+x_0)^2}\right]^{\frac{1}{3}} y, \tag{39}$$

where $-x_0 = O(R)$ is the 'virtual origin' of the jet. The exact value of x_0 depends on the initial profile on $x = 0$. The velocity profile corresponding to (39) is

$$u(x,y) = \left(\frac{3k^2}{32}\right)^{\frac{1}{3}} \left(\frac{R}{x+x_0}\right)^{\frac{1}{3}} \operatorname{sech}^2\left[\frac{kR^2}{48(x+x_0)^2}\right] y. \tag{40}$$

It is a lucky coincidence that one of the self-preserving profiles considered in the inviscid analysis in §2 has a similar 'sech²' dependence on the transverse coordinate y. It is for this reason that the solution given by Moreau (1963*a*, *b*) is possible. (It may be remarked that there is no self-preserving solution for the problem of the two-dimensional jet in an *aligned* field.)

The following derivation of the relevant similarity solution differs from that given by Moreau in certain respects. The simple Lagrangian form of the solution when $\nu = 0$ suggests that we use (x, ψ) as independent variables instead of (x, y) (the 'von Mises transformation').† Since

$$\left(\frac{\partial u}{\partial x}\right)_y = \left(\frac{\partial u}{\partial x}\right)_\psi - v\left(\frac{\partial u}{\partial \psi}\right)_x, \quad \left(\frac{\partial u}{\partial y}\right)_x = u\left(\frac{\partial u}{\partial \psi}\right)_y,$$

equation (15) takes the form

$$\frac{\partial u}{\partial x} = -N + R^{-1}\frac{\partial}{\partial \psi} u \frac{\partial u}{\partial \psi}. \tag{41}$$

Let us look for a similarity solution of the form

$$u(x,\psi) = u_1(x) f(\eta), \quad \eta = \psi/Q(x). \tag{42}$$

Here $u_1(x)$ is the velocity on $\psi = 0$ (i.e. on $y = 0$) so that $f(0) = 1$. Since ψ varies from $-Q$ to Q as y varies from $-\infty$ to ∞, the range of η is -1 to 1. Substitution of (42) in (41) gives

$$u_1' f - \frac{u_1 Q'}{Q}\eta f' = -N + R^{-1}\frac{u_1^2}{Q^2}(ff')'. \tag{43}$$

Now suppose that f can be expanded in Taylor series

$$f = \sum_{n=0}^{\infty} a_{2n}\eta^{2n} \quad (a_0 = 1), \tag{44}$$

convergent in some neighbourhood of $\eta = 0$. The series contains only even powers of η since the velocity profile is symmetrical. Then

$$\left.\begin{aligned} ff' &= \sum_{n=1}^{\infty} C_n \eta^{2n+1}, \\ \text{where} \qquad C_n &= \sum_{r=0}^{n} 2(n+1-r)\, a_{2r}\, a_{2(n+1-r)}. \end{aligned}\right\} \tag{45}$$

† A similar approach has been explored by Moreau (private communication).

On $\eta = 0$, equation (43) becomes

$$u_1' = -N + 2a_2 u_1^2/RQ^2; \tag{46}$$

moreover, equating the coefficients of η^{2n} in (43) gives

$$a_{2n}\left(\frac{u_1'}{u_1} - 2n\frac{Q'}{Q}\right) = (2n+1)\,C_n \frac{u_1}{RQ^2} \quad (n = 1, 2, 3, \ldots). \tag{47}$$

These equations are clearly independent, and since there are only two functions $u_1(x)$ and $Q(x)$ at our disposal, we can satisfy only one of the equations (47) in addition to (46). Hence the coefficients a_{2n} must vanish for $n = 2, 3, \ldots$† (and then from (45), $C_n = 0$ for $n = 3, 4, \ldots$). Hence $f = 1 + a_2\eta^2$. Since the series (44) terminates, it is (trivially) convergent throughout the range $|\eta| \leqslant 1$, and the condition $f(1) = 0$ implies $a_2 = -1$, i.e.

$$f(\eta) = 1 - \eta^2. \tag{48}$$

As might be expected, this corresponds to the 'sech²' profile when (42) is expressed in terms of (x, y).

Equation (47), with $n = 1$, now integrates to give

$$\frac{u_1}{Q^2} = \frac{R}{6(x+x_0)}, \tag{49}$$

where x_0 is a constant of integration, and (46) then integrates to give

$$u_1(x) = \frac{CR^{\frac{1}{3}}}{(x+x_0)^{\frac{1}{3}}} - \tfrac{3}{4}N(x+x_0), \tag{50}$$

where C is a further constant of integration.

Comparing (50) with (40) shows that $C = (3k^2/32)^{\frac{1}{3}}\ (= O(1))$, and that magnetic forces are in fact negligible for $(x+x_0) \ll R^{\frac{1}{4}}N^{-\frac{3}{4}}$, or, remembering that $x_0 = O(R)$ and that $M^2 = RN \ll 1$, for

$$x \ll R^{\frac{1}{4}}N^{-\frac{3}{4}}. \tag{51}$$

The flux in the jet for $x \gg x_0$ is given from (49) by

$$Q(x) = \left(\frac{6}{R}\right)^{\frac{1}{2}} (CR^{\frac{1}{3}}x^{\frac{2}{3}} - \tfrac{3}{4}Nx^2)^{\frac{1}{2}}. \tag{52}$$

This reaches a maximum at

$$x = 0{\cdot}545\, C^{\frac{3}{4}}R^{\frac{1}{4}}N^{-\frac{3}{4}} = x_m, \quad \text{say}, \tag{53}$$

and falls to zero at the 'stopping plane',

$$x = 1{\cdot}24 C^{\frac{3}{4}}R^{\frac{1}{4}}N^{-\frac{3}{4}} = 1{\cdot}24 C^{\frac{3}{4}}M^{\frac{1}{2}}N^{-1} \approx 2{\cdot}28 x_m = x_s \quad \text{say}. \tag{54}$$

This is to be contrasted with the corresponding result $x_s \approx N^{-1}$ valid in the inviscid limit $M \gg 1$ (§3). When $M \ll 1$, the result (54) clearly indicates the extent to which viscous forces can modify the annihilation process.

† We must have $a_2 \neq 0$, since otherwise, equations (47) successively imply $a_4 = 0$, $a_6 = 0, \ldots$, and the condition $f(1) = 0$ cannot be satisfied.

Again, boundary-layer theory breaks down as the stopping plane $x = x_s$ is approached. It may easily be verified that $|v(x,\infty)| = |dQ/dx|$ is small compared with $u_1(x)$ only if

$$1 - x/x_s \gg N^{\frac{1}{6}}R^{-\frac{1}{2}}; \tag{55}$$

when $x/x_s = 1 - O(N^{\frac{1}{6}}R^{-\frac{1}{2}})$, pressure forces again modify the final stage of annihilation, in a manner whose elucidation requires analysis of the full Navier–Stokes equations.

A sketch of the streamline pattern determined by the solution discussed in this section has been given by Moreau (1963*a*, *b*) and by Craya & Moreau (1964).

4. Speculations on further physical effects associated with jet annihilation

4.1. *The nature of the flow near and beyond the stopping plane*

The fact that a jet is stopped by a transverse field raises the question of what happens in the neighbourhood of the stopping plane $x = x_s$. The jet is clearly split into two 'deflected' jets alined in the $\pm y$ directions, and centred approximately on $x = x_s$. The momentum in each of these deflected jets is of the same order as the momentum in the original jet at the section at which boundary-layer theory begins to break down, i.e. at $x = x_s(1 - O(N^{-\frac{1}{2}}))$ for the top-hat inviscid jet ($M^2 \gg 1$) and at $x = x_s(1 - O(NR^{-\frac{1}{2}}))$ for the viscous (Moreau) jet ($M^2 \ll 1$). This momentum is

$$F_1 \approx \begin{cases} N^{\frac{1}{2}}F_0 & (M^2 \gg 1), \\ N^{\frac{1}{4}}R^{-\frac{3}{4}}F_0 = M^{\frac{1}{2}}R^{-1}F_0 & (M^2 \ll 1). \end{cases} \tag{56}$$

In either case, $F_1 \ll F_0$, so that the momentum in the deflected jet is small compared with the initial momentum in the primary jet.

In some respects, the strictly inviscid jet is simplest to visualize (figure 2). There is no further magnetic resistance to the flow after deflexion, and the deflected jets can flow parallel to the field $\mathbf{B}_0$ without any complications associated with viscous entrainment.

In the case of viscous deflexion however, it seems likely that the streamline pattern should be qualitatively as indicated in figure 3. For $y \gg x_s$, the flow presumably settles down to a weak wall-jet in the y-direction with constant flux Q_0. At a sufficient distance, inertia forces must be negligible, and the flow is controlled by the balance of magnetic and viscous forces. The pattern is again of course symmetrical about $y = 0$.

The eddies in figure 3 are generated by viscous entrainment, which is particularly strong near the slit. The length scale in the y-direction, y_c say, of the small eddy may be crudely estimated by the following argument. The deflected jet is approximately centred at $x = x_s$. It is a low momentum jet for $M^2 \ll 1$ (equation 56), and at sufficient distance from the plane $y = 0$, inertia forces are certainly negligible. The spread of the deflected jet is then described by the equation

$$\left(\frac{\partial^2}{\partial y^2} - M\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial y^2} + M\frac{\partial}{\partial x}\right)\psi = 0. \tag{57}$$

The relevant solutions depend on the similarity variable $M^{\frac{1}{2}}(x-x_s)/y^{\frac{1}{2}}$, i.e. the spread is parabolic. The jet interacts with the wall $x = 0$ when

$$y \approx y_c = Mx_s^2 = R/N \gg 1. \tag{58}$$

Note that

$$y_c/x_s = R^{\frac{3}{4}}N^{-\frac{1}{4}} \gg 1, \tag{59}$$

so that the eddy is extended in the y-direction.

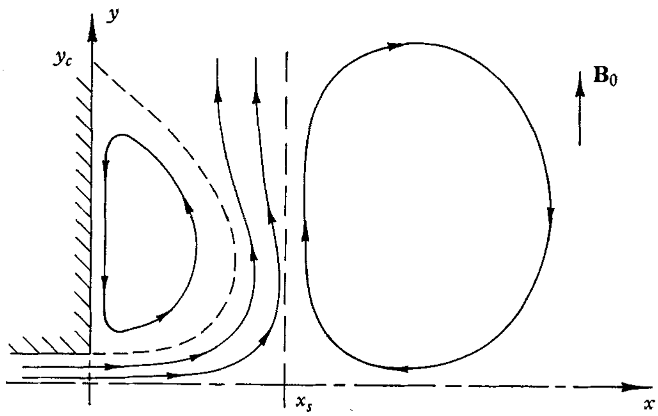


FIGURE 3. Qualitative sketch of the probable streamline pattern when $M^2 \ll 1$. The eddies are generated by viscous entrainment.

4.2. *The effect of the presence of rigid boundaries at* $y = \pm y_0$

The foregoing analysis is somewhat unrealistic in that the effect of the boundaries $y = \pm y_0$ in the region $x > 0$ has been totally neglected. It is to be hoped that such neglect is legitimate, at any rate in certain regions of the flow, when $y_0 \gg 1$. However, it is now possible to predict at least qualitatively what the effect of rigid boundaries at $y = \pm y_0$ will be, even when y_0 is not large. There are essentially two possibilities when $N \ll 1$, $M^2 \gg 1$.

(i) $N^{\frac{1}{2}} \ll y_0^{-1}$ (figure 4*a*). In this case, an essentially inviscid jet emerges from the slit and begins to be influenced by the walls $y = \pm y_0$ at a distance

$$x_0 \approx N^{-1}(1-y_0^{-1}) \tag{60}$$

downstream. A Hartmann profile is established for $x \gg x_0$. Weak viscous entrainment will generate corner eddies as indicated.

(ii) $1 \gg N^{\frac{1}{2}} \gg y_0^{-1}$ (figure 4*b*). In this case, the jet is stopped at $x_0 \approx N^{-1}$ before it begins to interact with the walls $y = \pm y_0$. As N increases, the region of jet-type flow shrinks in extent. Although the analysis does not permit us to increase N to order unity and larger, it is not difficult to predict what happens in this third case.

(iii) $N \gg 1$ (figure 4*c*). Here the jet is extinguished as soon as it emerges from the slit, and there is a narrow region of adjustment from one Hartmann profile (for $x < 0$) to the other (for $x > 0$). This situation is related to that considered by Hunt & Leibovich (1967) who studied Hartmann flow through a channel of varying cross-section under the conditions $M \gg 1$, $N \gg 1$. They were able to analyse the sudden adjustment of a Hartmann profile due to a sudden change of *gradient* of the channel walls—in which case the thickness of the adjustment region is $O(N^{-\frac{1}{3}})$—but were unable to analyse the type of situation considered here in which the cross-sectional *area* suddenly changes. In the region of adjustment, $\nabla^2 \approx \partial^2/\partial x^2$, and consideration of the inertia-magnetic balance in (8) again suggests that the region of adjustment has thickness $O(N^{-\frac{1}{3}})$ as indicated in figure (4*c*).

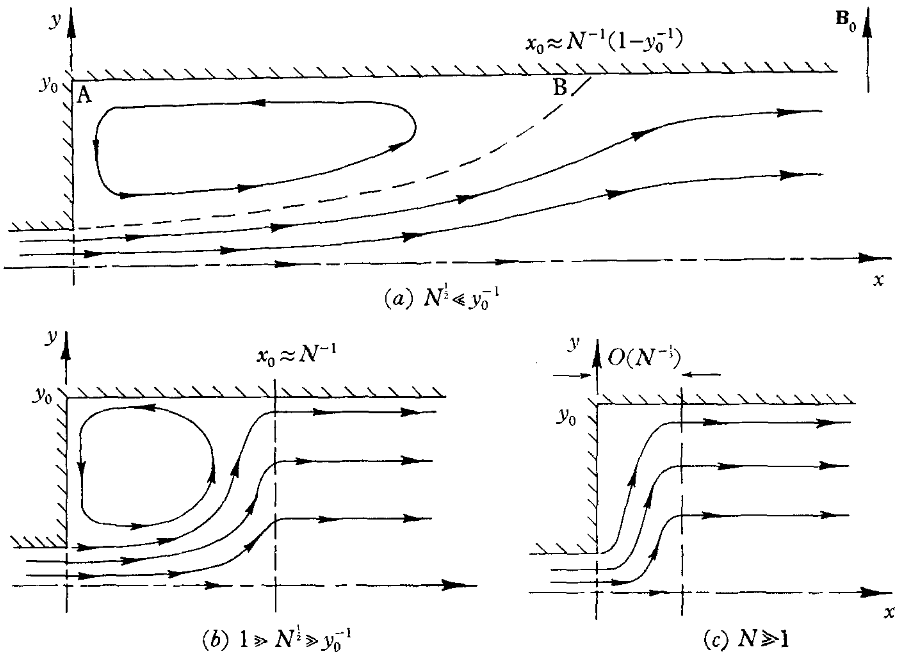


FIGURE 4. Qualitative sketch of the expansion of Hartmann flow with $M^2 \gg 1$ at a sudden increase in channel width.

In the case $M^2 \ll 1$, the width of the Moreau jet, from (50) and (52) is

$$\delta(x) \approx \left(\frac{6}{R}\right)^{\frac{1}{2}} \frac{x^{\frac{2}{3}}}{(CR^{\frac{1}{3}} - \frac{3}{4}Nx^{\frac{4}{3}})^{\frac{1}{2}}}, \tag{61}$$

and this becomes comparable with y_0 when

$$y_0^2(CR^{\frac{1}{3}} - \tfrac{3}{4}Nx^{\frac{4}{3}}) \approx \frac{6}{R}x^{\frac{4}{3}},$$

i.e. at

$$x \approx x_0 = \left[\frac{CR^{\frac{1}{3}}y_0^2}{(6/R) + \frac{3}{4}Ny_0^2}\right]^{\frac{3}{4}}. \tag{62}$$

There are three further possibilities, each giving rise to a distinctive pattern of flow:

(iv) $y_0 \ll M^{-1}$; in this case $x_0 \ll x_s$, and the primary jet interacts with the walls $y = \pm y_0$ before it is stopped (cf. figure 4*a*).

(v) $M^{-1} \ll y_0 \ll M^{-1}(R^3/N)^{\frac{1}{4}}$; in this case the jet is stopped before it interacts with the walls $y = \pm y_0$, but the deflected jets meet these walls before they interact with the wall $x = 0$ (cf. figure 4*b*).

(vi) $y_0 \gg M^{-1}(R^3/N)^{\frac{1}{4}}$, or equivalently $y_0 \gg y_c$; in this case, the deflected jets interact with the wall $x = 0$ before they meet the walls $y = \pm y_0$ (cf. figure 3).

A situation similar to that depicted in figure 4 has been studied experimentally by Moreau (1966*a*, *b*). The values of the parameters R and y_0 (in the notation of the present paper) were 2×10^3 and 25 respectively. The Hartmann number M varied in the range $1 \lesssim M \lesssim 3$. Qualitatively, the jet annihilation effect was demonstrated. It is not possible to make quantitative comparison between the theory, justifiable only when $M^2 \ll 1$ or $M^2 \gg 1$, and the experiments for which

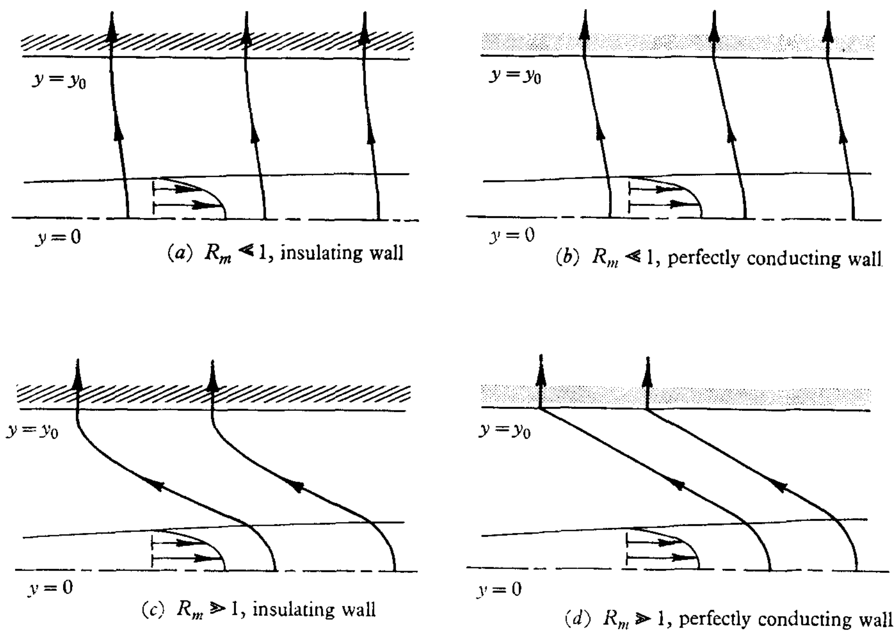


FIGURE 5. The lines of force of the magnetic field when $R_m \ll 1$ and when $R_m \gg 1$. In (*a*) and (*c*) the wall $y = y_0$ is insulating, and the current returns through the fluid outside the jet. In (*b*) and (*d*), the wall $y = y_0$ is perfectly conducting, and it conducts the return current.

$M^2 = O(1)$. However, one feature of the experiments is a little unexpected. The extent AB (figure 4*a*) of the region of closed streamlines should vary approximately as N^{-1} when $M^2 \gg 1$ and as $N^{-\frac{2}{3}}$ when $M^2 \ll 1$. The three streamline patterns inferred by Moreau from probe measurements suggest a dependence nearer to $N^{-\frac{1}{3}}$. Some further investigation over a much wider range of values of M would clearly be valuable.

4.3. *The effect of increasing R_m*

The lines of force of the total magnetic field (i.e. including the perturbation due to the currents in the fluid) are sketched in figures 5*a* and 5*b* in the two cases (*a*) in which the boundaries $y = \pm y_0$ are insulating and (*b*) in which they are perfectly conducting electrodes. In case (*a*), the current J induced in the jet returns through the fluid outside the jet, and in case (*b*) it returns in current sheets on the electrodes $y = \pm y_0$. In either case the maximum value of $|B_y|/B_0$ is $O(R_m)$, and this has so far been assumed small.

If we now allow R_m to increase to order unity and larger, $|B_y|/B_0$ will certainly increase (although not necessarily in simple proportion to R_m), and it seems likely that the lines of force in the two cases will then be as sketched in figures (5c) and (5d). The induced component B_y of the field does not affect the streamwise component of the jet, although it must affect the entrainment; it seems likely that the entrainment will be along the magnetic lines of force, i.e. that $\mathbf{u}$ and $\mathbf{B}$ will be parallel outside the jet (and of course outside boundary layers on any solid boundaries).

The important result that a transverse field annihilates a jet cannot be affected by increasing R_m, although the 'stopping distance' x_s may now depend on R_m as well as on N (and R also in the viscous case). Again for $x \gg x_s$, the flow will asymptotically approach a Hartmann profile.

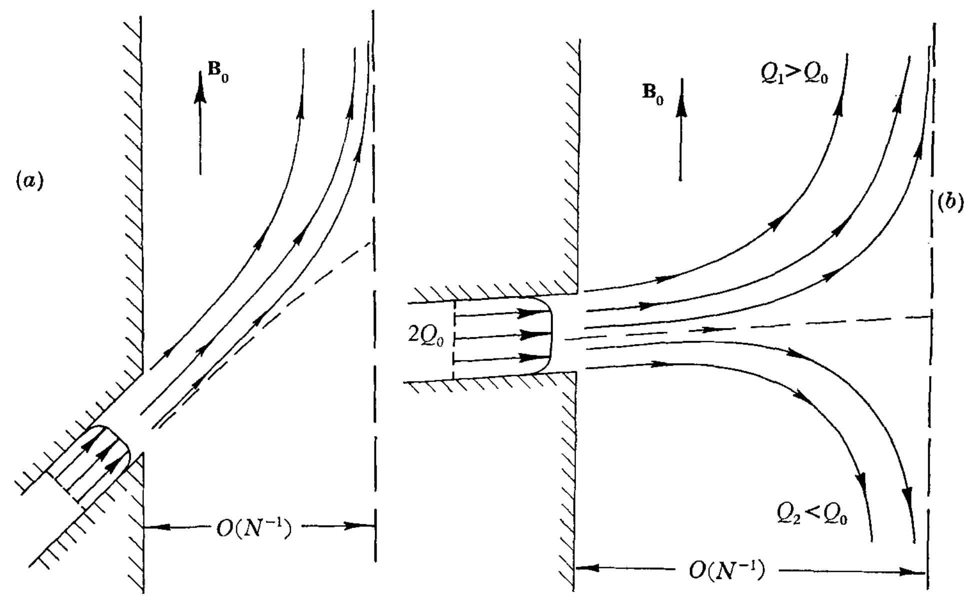


FIGURE 6. Oblique injection of an inviscid jet into a transverse field. The parameter N is defined in terms of the x-component of velocity at the slit.

4.4. *Deflexion of a jet oblique to the applied field*

So far it has been assumed that conditions are symmetrical about the plane $y = 0$. However, if the jet, on emerging from the slit $|y| < 1$, $x = 0$, has a net momentum in (say) the positive y-direction, then this momentum is conserved, and the jet may be almost totally deflected in this direction (figure 6a). This behaviour resembles that of a jet directed into a transverse wind, although in that case, the jet penetrates an infinite distance in the x-direction (on inviscid analysis) (Taylor 1954), whereas in the magnetic problem considered here, it penetrates the same finite distance x_s as for symmetrical injection.

If the initial transverse momentum is small, then the jet may be still deflected into two jets, but the deflected jet in the negative y-direction will have a smaller momentum than that in the positive y-direction (figure 6b).

It is evident that the solution of the inviscid form of equation (14), viz.

$$N\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}, \tag{63}$$

subject to the condition that $\partial\psi/\partial y$ is prescribed on $x = 0$, is not unique, but physical considerations suggest that it can be rendered unique if the initial transverse momentum

$$F_t = -\int_{-\infty}^{\infty}\left(\frac{\partial\psi}{\partial x}\right)_{x=0} dy \tag{64}$$

is also prescribed.

4.5. *Free jet in a transverse field*

The behaviour of a free jet of conducting fluid such as mercury into a non-conducting environment such as air in the presence of a transverse field is very different. Suppose again that the jet is approximately two-dimensional, that it emerges from a slit on the wall $x = 0$, and that it is bounded by the surfaces $y = \pm y_0(x)$. Suppose also that conditions at $z = \pm z_0$ are such that no net current can flow in the z-direction, e.g. the jet might be bounded at $z = \pm z_0$ by the same non-conducting medium (e.g. air) that bounds it at $y = \pm y_0(x)$. Then any current j_z in the jet, must return through the jet, i.e. the total current

$$J = \int_{-y_0(x)}^{y_0(x)} j_z\, dy$$

must vanish. This condition determines E_z. Moreover, since the medium for $|y| > y_0(x)$ is non-conducting, there need be no pressure gradient, since there is no electromagnetic force in this region, i.e. $\partial p/\partial x = 0$. The (dimensionless) expression for the current becomes simply

$$j_z = N(u-\overline{u}), \tag{65}$$

where $\overline{u}(x)$ $(=Q/y_0(x))$ is the average velocity at the section x, and the inviscid equation of motion is

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -N(u-\overline{u}). \tag{66}$$

Clearly if $u = \overline{u}$, i.e. if u is constant across the jet cross-section, then there is no Lorentz force, and the inviscid jet is unaffected by the presence of the magnetic field. If $u \neq \overline{u}$ at any section, then the Lorentz force is such as to accelerate the fluid where $u < \overline{u}$ and to decelerate it where $u > \overline{u}$, i.e. it tends to smooth out variation in the velocity across the jet. The Lagrangian solution of (66), analogous to (18), shows that differences in velocity $u-\overline{u}$ at any section $x = x_0$ are eliminated in a (dimensional) distance of order $\rho(u-\overline{u})/\sigma B_0^2$ downstream of x_0. Viscous forces again have negligible effect if the Hartmann number based on the slit width is large.

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